MATCHING THE PARAMETERS OF SHELLTURBULENCE MODELS WITH THE PROBABILITIES OF INTERACTION

СОГЛАСОВАНИЕ ПАРАМЕТРОВ КАСКАДНЫХ МОДЕЛЕЙ ТУРБУЛЕНТНОСТИ С ВЕРОЯТНОСТЯМИ ВЗАИМОДЕЙСТВИЯ ВОЛНОВЫХ ОБОЛОЧЕК

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MHD equations for incompressible fluid

Description in physical space – fields $\mathbf{v}(\mathbf{x},t)$, $\mathbf{B}(\mathbf{x},t)$, $p(\mathbf{x},t)$, $\mathbf{f}(\mathbf{x},t)$

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{v} = -\nabla p + \mathrm{Re}^{-1}\Delta \mathbf{v} + (\nabla \times \mathbf{B}) \times \mathbf{B} + \mathbf{f}, \quad \nabla \mathbf{v} = 0,$$

where Re – Reynolds number.

Equation of magnetic field induction

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \mathrm{Re}_m^{-1}\Delta \mathbf{B}, \quad \nabla \mathbf{B} = 0,$$

where Re_m – magnetic Reynolds number.

Description in Fourier space – fields $\hat{\mathbf{v}}(\mathbf{k},t)$, $\hat{\mathbf{B}}(\mathbf{k},t)$, $\hat{\mathbf{f}}(\mathbf{x},t)$

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} = \imath \int_{\mathbb{R}^3} d\mathbf{q} \int_{\mathbb{R}^3} d\mathbf{s}\, \delta(\mathbf{k}+\mathbf{q}+\mathbf{s})\, S(\mathbf{k},\mathbf{q},\mathbf{s}) \bullet \hat{\mathbf{v}}^*(\mathbf{q}) \bullet \hat{\mathbf{v}}^*(\mathbf{s}) - \mathrm{Re}^{-1}k^2\hat{\mathbf{v}} +$$

$$+\imath \int_{\mathbb{R}^3} d\mathbf{q} \int_{\mathbb{R}^3} d\mathbf{s}\, \delta(\mathbf{k}+\mathbf{q}+\mathbf{s})\, L(\mathbf{k},\mathbf{q},\mathbf{s}) \bullet \hat{\mathbf{B}}^*(\mathbf{q}) \bullet \hat{\mathbf{B}}^*(\mathbf{s}) + \mathbf{k} \times \left(\mathbf{k} \times \hat{\mathbf{f}}\right)/k^2, \quad \mathbf{k}\cdot\hat{\mathbf{v}} = 0$$

$$\frac{\partial \hat{\mathbf{B}}}{\partial t} = \imath \int_{\mathbb{K}^3} d\mathbf{q} \int_{\mathbb{R}^3} d\mathbf{s}\, \delta(\mathbf{k}+\mathbf{q}+\mathbf{s})\, W(\mathbf{k},\mathbf{q},\mathbf{s}) \bullet \hat{\mathbf{v}}^*(\mathbf{q}) \bullet \hat{\mathbf{B}}^*(\mathbf{s}) - \mathrm{Re}_m^{-1}k^2\hat{\mathbf{B}}, \quad \mathbf{k}\cdot\hat{\mathbf{B}} = 0$$

where $S(\cdot,\cdot,\cdot)$, $L(\cdot,\cdot,\cdot)$, $W(\cdot,\cdot,\cdot)$– some real tensor functions of rank 3.

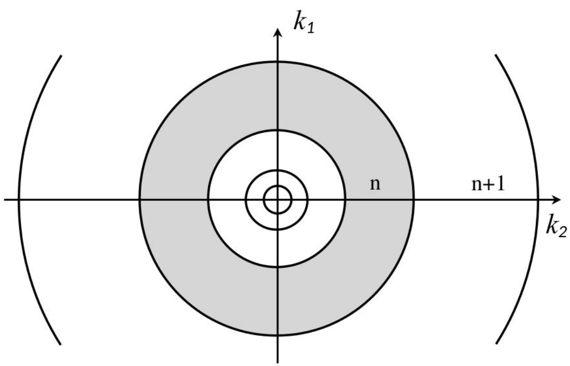
Shells in Fourier space

Let D be the linear size of the turbulent system (unit of length) and the number $q > 1$.

Introducing hierarchical scale ranges $D_n = (q^{-n-1}; q^{-n}]$, where $n \in \mathbb{Z}$.

The sizes of the ranges are $(q-1)/q^n$, and $\bigcup_{n=-\infty}^{+\infty} D_n = (0; +\infty)$.

Corresponding shells $P_n = \{\mathbf{k} \mid q^n \le \|\mathbf{k}\| < q^{n+1}\}$



We introduce complex collective variables of velocity $U_n(t)$ and magnetic field $B_n(t)$, where $|U_n(t)|$ and $|B_n(t)|$ are interpreted as measures of all structures (vortices) of a given range of wave numbers .

For example, $|U_n(t)| \sim \int_{\mathbf{k} \in P_n} \|\hat{\mathbf{v}}(\mathbf{k}, t)\| \, d\mathbf{k}$.

Shell models

Shell turbulence model – a system of dynamic equations for collective variables.

The structure of the equations should be similar to the structure of MHD equations in Fourier space

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} = \imath \int_{\mathbb{R}^3} d\mathbf{q} \int_{\mathbb{R}^3} d\mathbf{s}\, \delta\left(\mathbf{k}+\mathbf{q}+\mathbf{s}\right) S\left(\mathbf{k},\mathbf{q},\mathbf{s}\right) \bullet \hat{\mathbf{v}}^*\left(\mathbf{q}\right) \bullet \hat{\mathbf{v}}^*\left(\mathbf{s}\right) - \mathrm{Re}^{-1} k^2 \hat{\mathbf{v}} +$$
$$+\imath \int_{\mathbb{R}^3} d\mathbf{q} \int_{\mathbb{R}^3} d\mathbf{s}\, \delta\left(\mathbf{k}+\mathbf{q}+\mathbf{s}\right) L\left(\mathbf{k},\mathbf{q},\mathbf{s}\right) \bullet \hat{\mathbf{B}}^*\left(\mathbf{q}\right) \bullet \hat{\mathbf{B}}^*\left(\mathbf{s}\right) + \mathbf{k} \times \left(\mathbf{k} \times \hat{\mathbf{f}}\right)/k^2, \quad \mathbf{k}\cdot\hat{\mathbf{v}} = 0$$
$$\frac{\partial \hat{\mathbf{B}}}{\partial t} = \imath \int_{\mathbb{K}^3} d\mathbf{q} \int_{\mathbb{R}^3} d\mathbf{s}\, \delta\left(\mathbf{k}+\mathbf{q}+\mathbf{s}\right) W\left(\mathbf{k},\mathbf{q},\mathbf{s}\right) \bullet \hat{\mathbf{v}}^*\left(\mathbf{q}\right) \bullet \hat{\mathbf{B}}^*\left(\mathbf{s}\right) - \mathrm{Re}_m^{-1} k^2 \hat{\mathbf{B}}, \quad \mathbf{k}\cdot\hat{\mathbf{B}} = 0$$

The most common class of shell models (models like GOY: Gledzer–Okhitani–Yamada)

$$\frac{dU_n}{dt} = \imath k_n \sum_{i,j=-\infty}^{+\infty} S_{ij} U^*_{n+i} U^*_{n+j} - \mathrm{Re}^{-1} k_n^2 U_n + \imath k_n \sum_{i,j=-\infty}^{+\infty} L_{ij} B^*_{n+i} B^*_{n+j} + f_n(t),$$
$$\frac{dB_n}{dt} = \imath k_n \sum_{i,j=-\infty}^{+\infty} W_{ij} U^*_{n+i} B^*_{n+j} - \mathrm{Re}_m^{-1} k_n^2 B_n,$$
$$n = -\infty, \ldots, +\infty, \quad S_{ij} = S_{ji}, \quad N_{ij} = N_{ji},$$

where $k_n = q^n$ is the wave number of the n-th shell, S_{ij}, L_{ij}, W_{ij} are real coefficients, $f_n(t)$ – models the external supply of energy to the n-th shell. Usually only $f_0(t) \neq 0$.

Within the same class, models differ from each other in matrices of nonlinear interactions S_{ij}, L_{ij}, W_{ij}.

Interaction restrictions

Limitation 1 – permissible range of interaction.

The interaction of shells is described by quadratic terms. The indices i and j define the «distance» between scales on a logarithmic scale.

A restriction on the «range» of interaction is introduced: $S_{ij} = L_{ij} = W_{ij} = 0$, if $|i| > P$ or $|j| > P$. If $P \leq 2$, then only neighboring shells interact. If $P > 2$, then the model is nonlocal (in scale space).

Limitation 2 – Interoperability.

The shells n, $n + i$, $n + j$ must contain the wave vectors from which a triangle is formed.

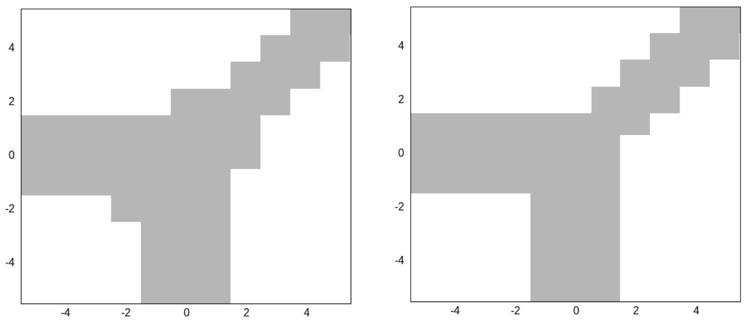


Figure for $P = 5$ and parameter $q = \left(1 + \sqrt{5}\right)/2$ – «golden ratio» (left) and $q \geq 2$ (right).

All coefficients outside the gray area are assumed to be zero.

For any wave numbers from $a \in P_{n-1}$ and $b \in P_n$ there is a wave number $c \in P_{n+1}$ such that $a + b = c$, if and only if $q = \left(1 + \sqrt{5}\right)/2$.

Interaction restrictions

Limitation 2 – the necessity of the existence of invariants.

For $f_n(t) \equiv 0$ and $\mathrm{Re}^{-1} = \mathrm{Re}_m^{-1} = 0$ (free motion of an inviscid ideally conducting medium), the MHD equations have quadratic invariants . The shell model must have their analogues.

- Total Energy

$$E = \frac{1}{2}\int_{\mathbb{R}^3} \left(\mathbf{v}^2 + \mathbf{B}^2\right)\, d\mathbf{x} = \frac{1}{16\pi^3}\int_{\mathbb{R}^3} \left(\|\hat{\mathbf{v}}\|^2 + \|\hat{\mathbf{B}}\|^2\right)\, d\mathbf{k} \sim \frac{1}{2}\sum_{n=-\infty}^{+\infty} \left(|U_n|^2 + |B_n|^2\right)$$

- Cross helicity

$$H_C = \int_{\mathbb{R}^3} \mathbf{v}\cdot\mathbf{B}\, d\mathbf{x} = \frac{1}{8\pi^3}\int_{\mathbb{R}^3} \hat{\mathbf{v}}\cdot\hat{\mathbf{B}}^*\, d\mathbf{k} \sim \sum_{n=-\infty}^{+\infty} \left(U_n B_n^* + U_n^* B_n\right)$$

- Squared magnetic field potential **A** (for two-dimensional flows)

$$A^2 = \int_{\mathbb{R}^3} \mathbf{A}^2\, d\mathbf{k} = \frac{1}{8\pi^3}\int_{\mathbb{R}^3} \left\|\frac{\imath}{k^2}\mathbf{k}\times\hat{\mathbf{B}}\right\|^2 d\mathbf{k} \sim \sum_{n=-\infty}^{+\infty} k_n^{-2}|B_n|^2$$

- Magnetic helicity (for three-dimensional flows)

$$H_B = \int_{\mathbb{R}^3} \mathbf{B}\cdot\mathbf{A}\, d\mathbf{k} = \frac{1}{8\pi^3}\int_{\mathbb{R}^3} \hat{\mathbf{B}}\cdot\left(\frac{-\imath}{k^2}\mathbf{k}\times\hat{\mathbf{B}}^*\right) d\mathbf{k} \sim \sum_{n=-\infty}^{+\infty} k_n^{-1}\mathrm{Re}B_n\mathrm{Im}B_n$$

Limitation 4 - conservation of phase volume.

For $f_n(t) \equiv 0$ and $\mathrm{Re}^{-1} = \mathrm{Re}_m^{-1} = 0$ the shell model should preserve the phase volume.

Magnetic helicity invariant problem

Magnetic helicity (for three-dimensional flows)

$$H_B = \frac{1}{8\pi^3}\int_{\mathbb{R}^3} \hat{\mathbf{B}} \cdot \left(\frac{-\imath}{k^2}\mathbf{k} \times \hat{\mathbf{B}}^*\right) d\mathbf{k} = \frac{1}{8\pi^3}\int_{\mathbb{R}^3} \frac{1}{k^2}\left[\mathbf{B} \cdot \left(-\imath\mathbf{k} \times \hat{\mathbf{B}}^*\right)\right] d\mathbf{k} \sim \imath \sum_{n=-\infty}^{+\infty} \frac{1}{k_n} B_n(t) B_n^*(t).$$

Energy and magnetic helicity invariants turn out to be incompatible in GOY-type models.

It is necessary to either change the type of invariant, or change the structure of the model equations.

Changing an invariant

$H_B \sim \sum_{n=-\infty}^{+\infty} (-1)^n k_n^{-1} |B_n|^2$ instead of $\sum_{n=-\infty}^{+\infty} k_n^{-1} \mathrm{Re} B_n \mathrm{Im} B_n$ Sometimes they take a more general

view $\hat{H}_B^\lambda = \sum_{n=-\infty}^{+\infty} (-1)^n k_n^{-\lambda} |B_n|^2$

In such forms, «magnetic helicity» is compatible with the energy in GOY-type models.

Conservation of cross-helicity

$$\frac{dH_C}{dt} = \sum_{n=-\infty}^{+\infty} \left(\frac{dU_n(t)}{dt} B_n^*(t) + \frac{dB_n^*(t)}{dt} U_n(t) + \frac{dU_n^*(t)}{dt} B_n(t) + \frac{dB_n(t)}{dt} U_n^*(t) \right) = 0,$$

where

$$\frac{du_n}{dt} = \imath k_n \sum_{i,j=-\infty}^{+\infty} \left(S_{ij} U_{n+i}^* U_{n+j}^* + L_{ij} B_{n+i}^* B_{n+j}^* \right), \quad \frac{dB_n}{dt} = \imath k_n \sum_{i,j=-\infty}^{+\infty} \left(W_{ij} U_{n+i}^* B_{n+j}^* \right),$$

$$U_n(t) = x_n(t) + \imath y_n(t), \quad B_n(t) = g_n(t) + \imath h_n(t),$$

It is clear that $\frac{dH_C}{dt}$ is a combination of trilinear forms from x_n, y_n, h_n, g_n.

We introduce the notations $n+i=l$, $n+j=m$, $q^n F_{ij} = F_{nlm}$, $q^n G_{ij} = G_{nlm}$, $q^n H_{ij} = H_{nlm}$, $q^n K_{ij} = K_{nlm}$ and calculate the monomials of the forms in Maple

```
> u[n]:=x[n]+I*y[n]:  B[n]:=g[n]+I*h[n]:
  u[l]:=x[l]+I*y[l]:  B[l]:=g[l]+I*h[l]:
  u[m]:=x[m]+I*y[m]:  B[m]:=g[m]+I*h[m]:
  du[n]:=I*S[n,l,m]*conjugate(u[l])*conjugate(u[m])+I*L[n,l,m]*conjugate(B[l])*conjugate
  (B[m]):
  dB[n]:=I*W[n,l,m]*conjugate(u[l])*conjugate(B[m]):
  Hc:=simplify(du[n]*conjugate(B[n])+u[n]*conjugate(dB[n])+conjugate(du[n])*B[n]+
  conjugate(u[n])*dB[n]);
```

$$Hc := 2 S\sim_{n,l,m} x\sim_l x\sim_m h\sim_n + 2 S\sim_{n,l,m} x\sim_l y\sim_m g\sim_n + 2 S\sim_{n,l,m} y\sim_l x\sim_m g\sim_n - 2 S\sim_{n,l,m} y\sim_l y\sim_m h\sim_n + 2 L\sim_{n,l,m} g\sim_l g\sim_m h\sim_n + 2 L\sim_{n,l,m} g\sim_l h\sim_m g\sim_n + 2 L\sim_{n,l,m} h\sim_l g\sim_m g\sim_n - 2 L\sim_{n,l,m} h\sim_l h\sim_m h\sim_n + 2 W\sim_{n,l,m} x\sim_n x\sim_l h\sim_m + 2 W\sim_{n,l,m} x\sim_n y\sim_l g\sim_m + 2 W\sim_{n,l,m} y\sim_n x\sim_l g\sim_m - 2 W\sim_{n,l,m} y\sim_n y\sim_l h\sim_m \quad (1)$$

Five forms with species monomials $x_n x_m h_l$, $x_n y_m g_l$, $g_n g_m h_l$, $h_n h_m h_l$ и $y_n y_l h_m$.

We reduce similar ones, taking into account permutations of indices, and equate the coefficients to zero.

Equations for coefficients dictated by invariants

Conditions for maintaining cross-helicity

$$L_{i,j} + L_{j,i} + 2^i L_{-i,j-i} + 2^i L_{j-i,-i} + 2^j L_{i-j,-j} + 2^j L_{-j,i-j} = 0$$

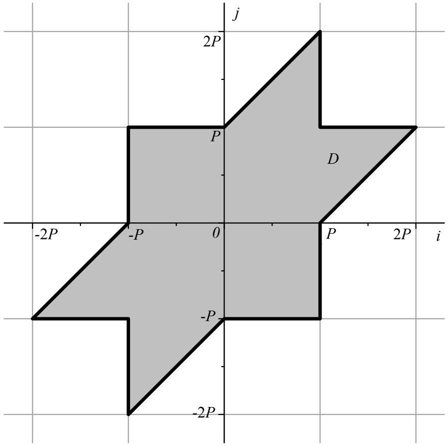
$$W_{i,j} + 2^j S_{i-j,-j} + 2^i W_{-i,j-i} + 2^j S_{-j,i-j} = 0$$

$$i, j = -\infty, \ldots, +\infty$$

Two infinite groups of homogeneous linear equations.

Similar conditions were obtained for the remaining invariants

- energy E – 2 groups of equations;
- magnetic helicity H_B – 2 groups of equations;
- squared magnetic field A^2 – 1 group of equations.



Pair (i, j) is an identifier of an equation within one group.

Taking into account the restriction on long-range action, only a finite number of equations do not degenerate into identities.

The stellar region D is distinguished by the condition

$$(|i| \le P \wedge |j| \le P) \vee (|i| \le P \wedge |i-j| \le P) \vee (|j| \le P \wedge |i-j| \le P),$$

It identifies a finite subsystem of equations that do not necessarily degenerate into identities.

Implementation of spectral laws

Let $W(k) \sim k^{\mu}$ be a characteristic of a stationary turbulent flow.
Then the total value of this characteristic in the n-th shell will be

$$W_n = S\left(q^n \leq k \leq q^{n+1}\right) = \int_{q^n}^{q^{n+1}} W(k)\, dk \sim \int_{q^n}^{q^{n+1}} k^{\mu}\, dk \sim q^{n(\mu+1)}.$$

For example, Kolmogorov's law for energy $E(k) \sim k^{-5/3}$, i.e. $E_n \sim q^{(-2/3)n}$.

In the shell model, the energy in the n-th shell is $|U_n|^2 + |B_n|^2$.

We require the existence of a stationary solution $U_n = |U_0|q^{-n/3}$ + $B_n = |B_0|q^{-n/3}$.

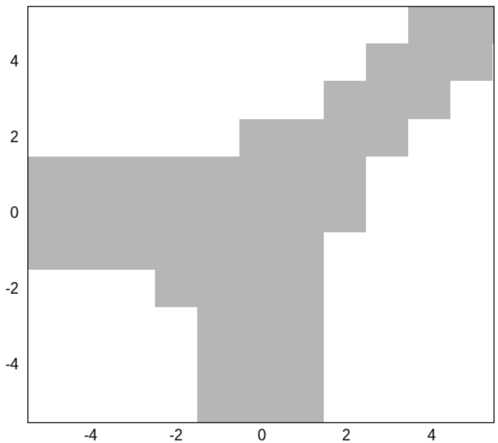
Let's substitute the solution into the model equations and get

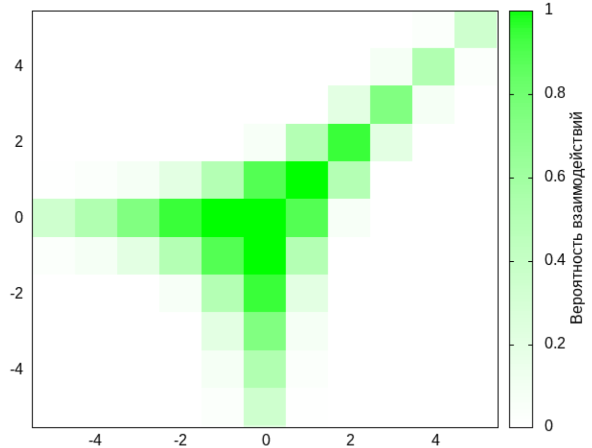
$$|U_0|^2 \sum_{i,j} \left(S_{ij} + Q_{ij} + R_{ij}\right) q^{-(i+j)/3} + |B_0|^2 \sum_{i,j} \left(L_{ij} + M_{ij} + N_{ij}\right) q^{-(i+j)/3} = 0,$$
$$\sum_{i,j} \left(F_{ij} + G_{ij} + H_{ij}\right) q^{-(i+j)/3} = 0.$$

The existence of such stationary solutions does not guarantee their stability.

Shell interaction probabilities and coefficients

The interaction of the n-th, $(n+i)$-th and $(n+j)$ shells is possible if it is possible to construct a triangle from segments with lengths from the intervals $[1; q]$, $[q^i; q^{i+1}]$, $[q^j; q^{j+1}]$. Then the Monte Carlo method can be used to calculate the probabilities p_{ij} of the interaction of waves from the shells.





On the left is the possibility of interaction, on the right is the probability of interaction for $q = \left(1+\sqrt{5}\right)/2$.

The coefficients of nonlinear terms in the models can be considered as some measures of the interaction of the n-th, $(n+i)$-th and $(n+j)$ shells. Therefore, it is reasonable to reconcile them with probabilities.

For example, let nonzero $L_{ij} = L_{ij}(\mathbf{s})$, $\mathbf{s} = [s_1, \ldots, s_k]^T$ is a vector of free parameters.
Reconciliation involves minimizing (in some sense) expressions

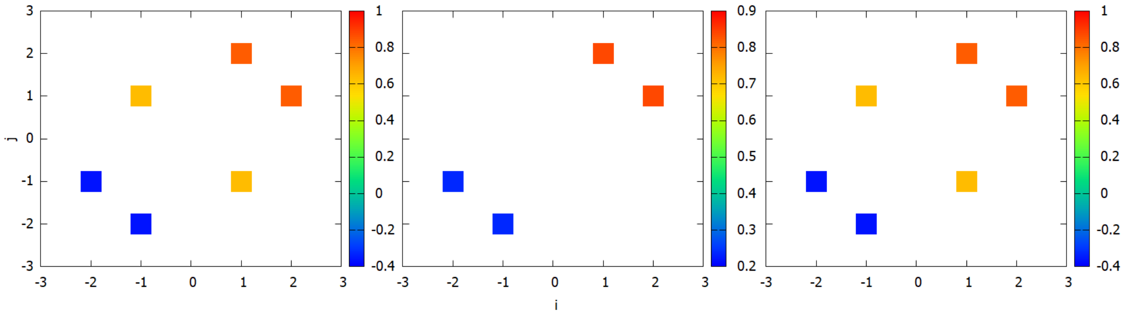
$$\sum_{i,j} \left[\left| \frac{|S_{ij}(\mathbf{s})| - p_{ij}}{p_{ij}} \right| + \left| \frac{|L_{ij}(\mathbf{s})| - p_{ij}}{p_{ij}} \right| + \left| \frac{|W_{ij}(\mathbf{s})| - p_{ij}}{p_{ij}} \right| \right] \to \min$$

And so on for all coefficients.

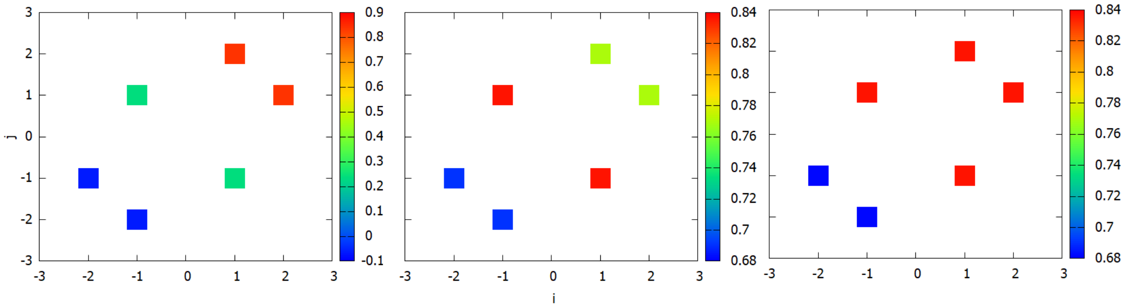
Computational experiments

Error calculation

$$osh = \sum_{i,j} \left| \frac{VER_{ij} - |S_{ij}|}{VER_{ij}} \right| + \left| \frac{VER_{ij} - |L_{ij}|}{VER_{ij}} \right| + \left| \frac{VER_{ij} - |W_{ij}|}{VER_{ij}} \right|.$$



Ratios of the resulting interaction coefficients to the interaction probabilities on a logarithmic scale (2D).



Ratios of the resulting interaction coefficients to the interaction probabilities on a logarithmic scale (3D).

Conclusions

- Previously, we developed approaches using computer algebra that allow us to obtain parametric classes of models that provide models of the necessary conservation laws and spectral laws;
- One formal method for selecting free parameters has been developed, in which the interaction coefficients are maximally consistent with the interaction probabilities;
- A comprehensive technology for constructing models with numerical values of parameters has been obtained. The generated shell models can be directly studied further by numerical methods.