Checking the stability of solutions in shell models using symbolic computation

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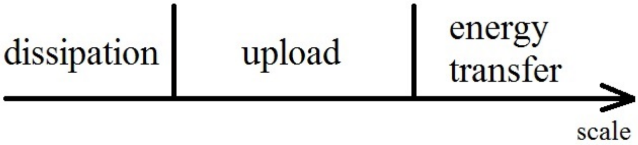
Turbulence representation



Physical space – processes are mixed, present at every point

Space of scales (wave vectors) – processes are separated

Shell turbulence models

The wavenumber axis is split into progressively expanding octaves

$$k_n <| \mathbf{k} |< k_{n+1}, k_n = 2^n k_0$$

General view of shell equations

$$\frac{d_t X_n}{dt} = \sum_{ij} Q_{nij} X_i X_j - K_n X_n + f_n \quad (1)$$

Q_{nij} – nonlinear interaction matrix
$K_n X_n \sim k_n^2 X_n$ – dissipative term
f_n – external forces

Table 1. Stored quantities

2D	3D
Conserved quantities in hydrodynamics	
Энстрофия $\Omega = \int (\mathrm{rot}\mathrm{v})^2 \, dV \sim \sum_n 2^{2n} u_n^2$	Спиральность гидродинамическая $H_V = \int (\mathrm{v} \cdot \mathrm{rot}\mathrm{v}) \, dV \sim \sum_n 2^n u_n^2$
Кинетическая энергия $E_V = \int \mathrm{v}^2 dV \sim \sum_n u_n^2$	Кинетическая энергия $E_V = \int \mathrm{v}^2 dV \sim \sum_n u_n^2$

Hydrodynamic system equations

GD turbulence equations in the Boussinesq approximation in dimensionless form

$$\frac{\partial \mathsf{v}}{\partial t} + (\mathsf{v}\bigtriangledown)\,\mathsf{v} = -\bigtriangledown p + Re^{-1} \bigtriangleup \mathsf{v}, \qquad (2)$$

Re – Reynolds number

Conserved quantities

Kinetic energy $E = \int \left(\mathsf{v}^2\right) dV$

Enstrophy $\Omega = \int (\mathrm{rot}\mathsf{v})^2 \, dV$

Hydrodynamic helicity $H_V = \int (\mathsf{v} \cdot \mathrm{rot}\mathsf{v})\, dV$

Shell model of hydrodynamic turbulence

Turbulent convection shell model including nonlocal interactions

$$\frac{d_t u_n}{dt} = \sum_{ij} S_{nij} u_i u_j - Re^{-1} k_n^2 u_n, \tag{3}$$

Conservation laws

Kinetic energy

$$\frac{dE}{dt} = 0 = \frac{d}{dt}\sum_n u_n^2 = 2\sum_n u_n \frac{du_n}{dt}$$

Enstrophy

$$\frac{d\Omega}{dt} = 0 = \frac{d}{dt}\sum_n 2^{2n} u_n^2 = 2\sum_n 2^{2n} u_n \frac{du_n}{dt}$$

Hydrodynamic helicity

$$\frac{dH_V}{dt} = 0 = \frac{d}{dt}\sum_n 2^n \left(-1\right)^n u_n^2 = 2\sum_n 2^n \left(-1\right)^n u_n \frac{du_n}{dt}$$

The resulting equations

The system consists of three families of equations for $s[i, j]$

$$\begin{gathered}(s_{i,j} + s_{j,i}) + 2^{i}\,(s_{-i,j-i} + s_{j-i,-i}) + 2^{j}\,(s_{i-j,-j} + s_{-j,i-j}) = 0,\\ (s_{i,j} + s_{j,i}) + 2^{3i}\,(s_{-i,j-i} + s_{j-i,-i}) + 2^{3j}\,(s_{i-j,-j} + s_{-j,i-j}) = 0,\\ (s_{i,j} + s_{j,i}) + (-1)^{i}2^{2i}\,(s_{-i,j-i} + s_{j-i,-i}) + (-1)^{j}2^{2j}\,(s_{i-j,-j} + s_{-j,i-j}) = 0\end{gathered} \tag{4}$$

The system is also complemented by symmetries of the following form

$$s_{i,j} - s_{j,i} = 0 \tag{5}$$

Scheme for calculating equations in a symbolic package

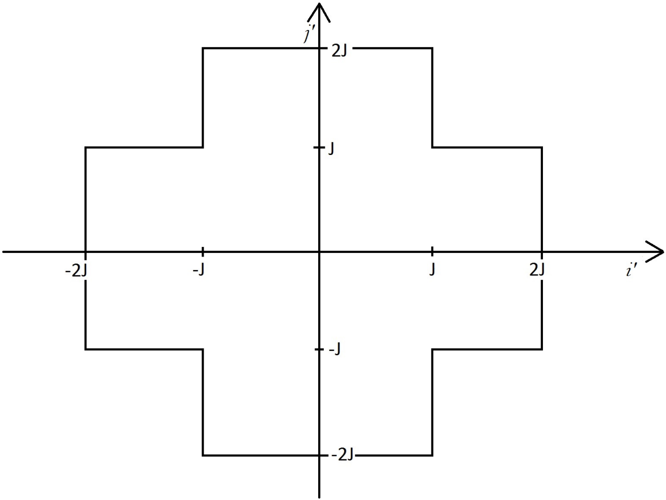


Рис. 1: Region of existence of equations.

Requirements for the existence of stationary solutions with a power-law dependence on the spatial scale

Stationary solutions

$$u = u_0 2^{\frac{p}{q}}, \tag{6}$$

Additional equations of the form are added to the system

$$\sum_{i,j} (s[i,j])\, 2^{-(i+j)\frac{p}{q}}, \tag{7}$$

where p/q – power law degree.

Interacting shells

Navier-Stokes equation in Fourier space

$$\frac{\partial \widehat{\mathrm{u}}}{\partial t} = i \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} S(\mathrm{k}, \mathrm{s}, \mathrm{q})\, \delta(\mathrm{s} + \mathrm{q} + \mathrm{k})\, \widehat{\mathrm{u}}(\mathrm{s}, t)\, \widehat{\mathrm{u}}(\mathrm{q}, t)\, d\mathrm{s} d\mathrm{q} - \\ -\nu \mathrm{k}^2 \widehat{\mathrm{u}}(\mathrm{k}, t) + \widehat{\mathrm{f}}(\mathrm{k}, t) \tag{8}$$

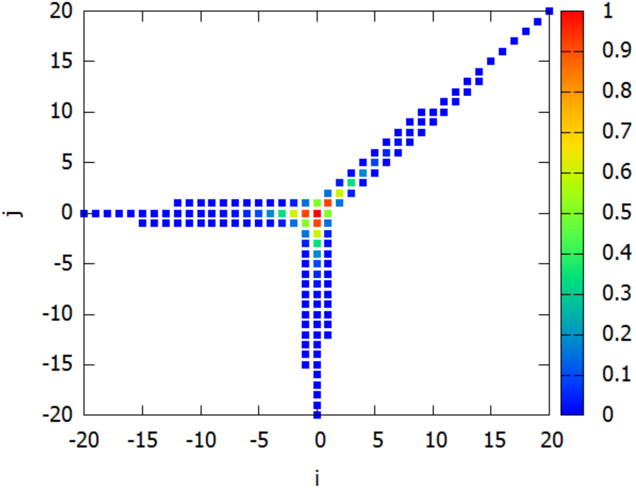
Ranges k, s и q

$$\begin{aligned} &\mathrm{k} \in \left(2^n; 2^{n+i}\right) \sim n, \\ &\mathrm{s} \in \left(2^{n+i}; 2^{n+i+1}\right) \sim n+i, \\ &\mathrm{q} \in \left(2^{n+j}; 2^{n+j+1}\right) \sim n+j, \end{aligned} \tag{9}$$

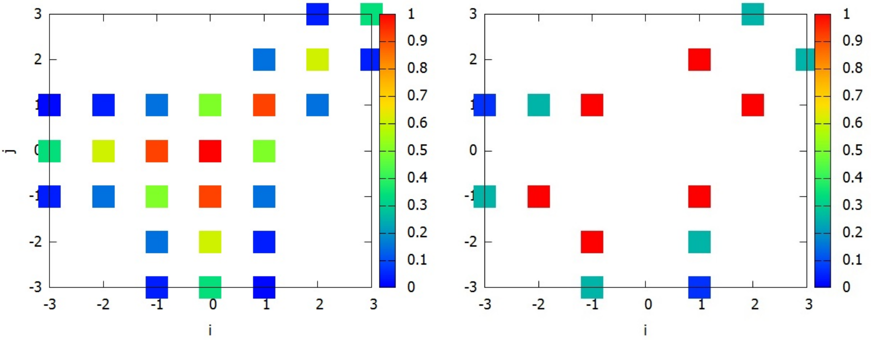
Triangle inequality

$$\begin{aligned} &2^n + 2^{n+i} \geqslant 2^{n+j}, \\ &2^{n+i} + 2^{n+j} \geqslant 2^n, \\ &2^n + 2^{n+j} \geqslant 2^{n+i}, \end{aligned} \tag{10}$$

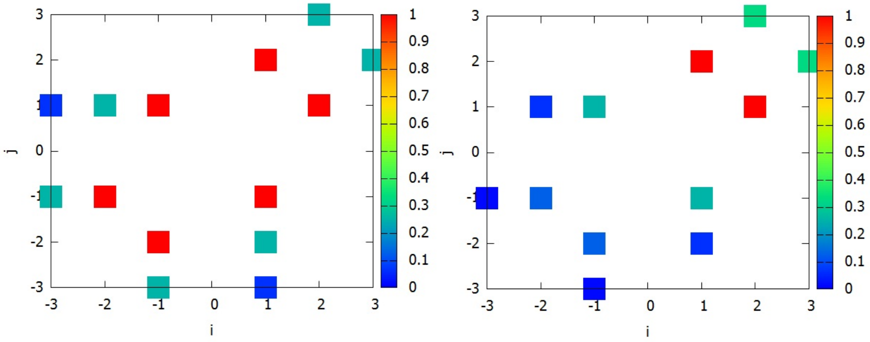
The probabilities of possible interactions of waves from the nth, (n+i)th, (n+j)th



Interaction probabilities subject to constraints for $J = 3$



Comparison with the intensities of interactions



Выводы

- The previously developed technology for constructing cascade models of computer algebra methods was supplemented and provides the presence of stationary solutions with any desired exponents. As a result, entire parametric classes of models are generated in an automated mode.

- The developed technique for identifying specific models in which the coefficients of nonlinear interactions would be maximum in magnitude to the probabilities of wave interactions of the corresponding scales.

- In fixed models, a study of power-law solutions for Lyapunov stability was realized. In the examples considered, these solutions turn out to be unstable.

- Further development of this technology presupposes the optimization of computational procedures, since the coordination of the coefficients with the probabilities is calculated for a very long time. And the agreement was not very good.