2-MODE $\alpha\Omega$-DYNAMO AS OSCILLATOR WITH MEMORY

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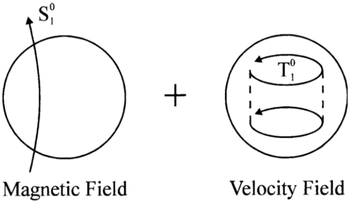
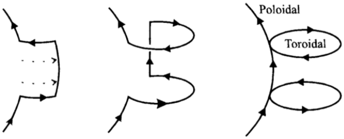
Mean field dynamo

Magnetic field $\mathbf{B} = \mathbf{B}' + \mathbf{b}$, velocity $\mathbf{v} = \mathbf{U} + \mathbf{u}$

$$\frac{\partial \mathbf{B}'}{\partial t} = rot\left(\mathbf{U} \times \mathbf{B}'\right) + rot\left\langle \mathbf{u} \times \mathbf{b} \right\rangle + \eta \Delta \mathbf{B}'$$

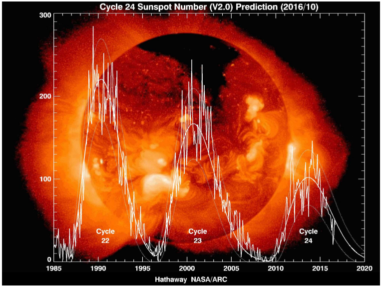
Parker's Dynamo



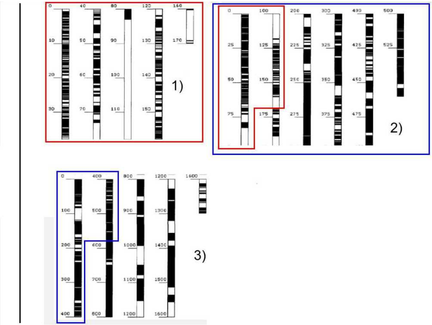


Cosmic Dynamo – Oscillating System

Cosmic dynamo system – is oscillator with chaotic components



Solar Cycle



Geomagnetic Polarity Scale
(1 – 170 Myr, 2 – 560 Myr, 3 – 1700 Myr)

Solar Cycle as a Strange Attractor

The simplest model of the nonlinear solar dynamo with chaotic regimes

$$\frac{dA}{dt} = -A + DB - CB,$$
$$\frac{dB}{dt} = -\sigma B + \sigma A,$$
$$\frac{dC}{dt} = -\nu C + AB,$$

where A and B – the azimutal componets of potential and magnetic field, C – deviation of helicity from its value in the absence of the magnetic field, σ – ratio of diffusion times of B and A, D – dynamo number.
Zeldovich, Ruzmaikin, 1980

The third equation under the condition $C(0) = 0$, is equivalent to

$$C(t) = \int_0^t e^{-\nu(t-\tau)} A(\tau) B(\tau)\, d\tau$$

This is dynamical quenching of the field by a t-parametric functional from helicity with an exponential kernel

2-mode $\alpha\Omega$-dynamo with stochastic memory

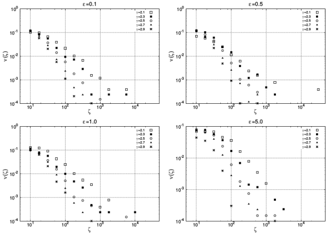
Distribution of polarity intervals

L. Feshchenko, G. Vodinchar // Nonlin. Processes Geophys., 2015

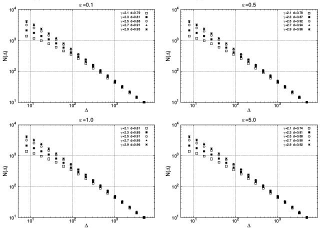
$$\frac{dB^T}{dt} = R_\Omega B^P - B^T, \quad \frac{dB^P}{dt} = R_\alpha B^T - B^P,$$

$$R_\alpha(t) = R_\Omega^{-1}\left[1 + \varepsilon\left(1 - B^2(t)\right) + \xi(t)\right],$$

where $\xi(t)$ is some non-Markov random pulse process with zero mean value.



Distribution of polarity intervals



Number of $N(\Delta)$ intervals of length Δ, which contain at least one inversion

Hausdorff dimensions for real geomagnetic polarity timescales for 170 Myr, 560 Myr, and 1700 Myr are 0.88, 0.83, and 0.87, respectively (Pechersky et al., 1997).

Basic equations

The $\alpha\Omega$-dynamo (Parker's dynamo) for axis-symmetric case:

$$\frac{\partial \mathbf{B}^T}{\partial t} = rot\,(\mathbf{v}^T \times \mathbf{B}^P) + \eta\Delta\mathbf{B}^T, \qquad (1)$$
$$\frac{\partial \mathbf{B}^P}{\partial t} = rot\,(\hat{\alpha}\mathbf{B}^T) + \eta\Delta\mathbf{B}^P.$$

$\mathbf{v}^T$ – toroidal large-scale velocity field (differential rotation);
$\hat{\alpha}$ – helicity of small-scale turbulence;
η – turbulent magnetic diffusivity.

The simplest 2-mode approximation:

$$\mathbf{B} = \mathbf{B}^T + \mathbf{B}^P = B^T(t)\mathbf{b}^T(\mathbf{r}) + B^P(t)\mathbf{b}^P(\mathbf{r}), \quad \|\mathbf{b}^T(\mathbf{r})\| = \|\mathbf{b}^P(\mathbf{r})\| = 1 \qquad (2)$$

Using the Galerkin method:

$$\int_\Omega \left[\mathbf{b}^T(\mathbf{r})\right]^2 d\mathbf{r} = \int_\Omega \left[\mathbf{b}^T(\mathbf{r})\right]^2 d\mathbf{r} = 1 \qquad (3)$$

$$\frac{dB^T}{dt} = \omega B^P - \eta^T B^T, \qquad (4)$$
$$\frac{dB^P}{dt} = \alpha B^T - \eta^P B^P.$$

The Galerkin coefficients: ω – is intensity of Ω-generator, α – is intensity of

Feedback

The influence of magnetic field on helicity: $\alpha = \alpha_0 - Q(B(t)^T, B(t)^P)$, where α_0 – value of alpha-effect in the absence of the strong magnetic field, and $Q(\cdot,\cdot)$ – is a quadratic form.

The energy of the field:

$$\int \mathbf{B}^2 \, d\mathbf{r} = |B^T(t)|^2 + |B^P(t)|^2.$$

The helicity of the field:

$$\int \mathbf{B} \, rot^{-1}\mathbf{B} \, d\mathbf{r} = B^T(t)B^P(t) \int \left[\mathbf{b}^T \, rot^{-1}\mathbf{b}^P + \mathbf{b}^P \, rot^{-1}\mathbf{b}^T \right] \, d\mathbf{r} \sim B^T(t)B^P(t).$$

We assume, that

$$Q = \int_0^t K(t-\tau)B^T(\tau)B^P(\tau) \, d\tau.$$

This predetermined expression Q are specify the model of feedback – hereditary quenching of α-effect by helicity.

$K(\cdot) \geq 0$ – some kernel, with the property $K(+\infty) = 0$.

2-modes model

The model equations:

$$\frac{dB^T}{dt} = \omega B^P - \eta^T B^T,$$

$$\frac{dB^P}{dt} = (\alpha_0 - Q)B^T - \eta^P B^P,$$

$$Q(t) = \int_0^t K(t-\tau)B^T(\tau)B^P(\tau)\,d\tau.$$

The model is closed by the initial conditions $B^T(0) = B_0^T, B^P(0) = B_0^P$. For planetary and stellar dynamo systems, it is reasonable to assume that $B_0^T = 0$.

$(B^T, B^P, Q) \equiv (0,0,0)$ – stationary point.
It is unstable if $D = \omega\alpha_0/(\eta^T\eta^P) > 1$, i.e. D – is dynamo-number.

Replacement of the variables:

$$t \to \eta^P t, \quad x(t) = B^T, \quad y(t) = \omega B^P/\eta^T, \quad z(t) = \omega Q/(\eta^T\eta^P).$$

New time scale – is the poloidal field diffusion time.

2-modes model

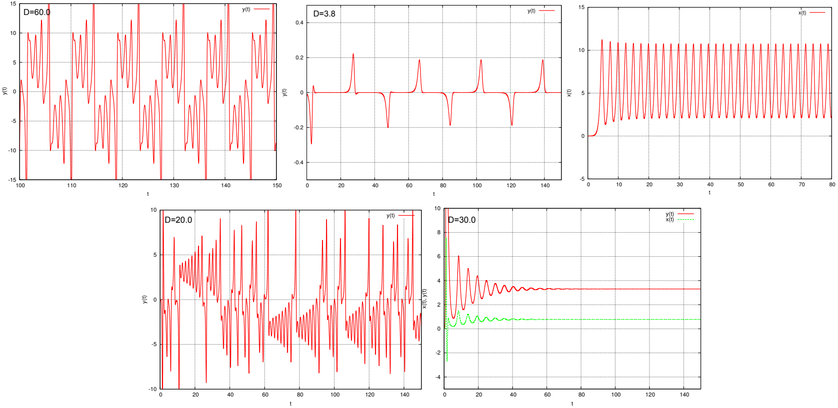
$$\frac{dx}{dt} = \sigma(y - x),$$
$$\frac{dy}{dt} = (D - z)x - y,$$
$$z(t) = \int_0^t K(t-\tau)x(\tau)y(\tau)\,d\tau,$$

where $\sigma = \eta^T/\eta^P > 1$. For Parker's dynamo $\sigma \approx 3.37$.
For all $x(0) = x_0$ and $y(0) = y_0$, this system equivalent to equation

$$\frac{d^2x}{dt^2} + (1+\sigma)\frac{dx}{dt} + \frac{K(0)}{2}x^3 - \left[\sigma(D-1) + \frac{x_0^2}{2}K(t) - w(t)\right]x = 0,$$
$$w(t) = \int_0^t J(t-\tau)x^2(\tau)\,d\tau,$$
$$J(\cdot) = \frac{K'(\cdot)}{2} + \sigma K(\cdot),$$

with initial conditions $x(0) = x_0, x'(0) = \sigma(y_0 - x_0)$.

Typical dynamic regimes



Hereditary oscillator

We further consider the case $x(0) = 0$.

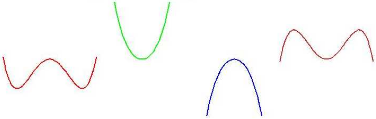
$$\frac{d^2x}{dt^2} + \underbrace{(1+\sigma)}_{>2}\frac{dx}{dt} + \underbrace{\frac{K(0)}{2}x^3 - [\sigma(D-1) - w(t)]\,x}_{\frac{\partial U}{\partial x}} = 0,$$

where

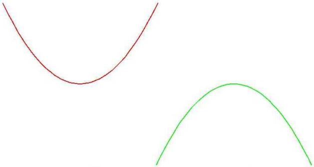
$$U = Ax^4 - B(t)x^2$$

$$A = K(0)/8, \quad 2B(t) = \underbrace{\sigma(D-1)}_{>0} - \underbrace{\int_0^t J(t-\tau)x^2(\tau)\,d\tau}_{w(t)}\,.$$

$K(0) \neq 0$ – instant feedback



$K(0) = 0$ – delay of feedback

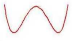


$K(s) = e^{-bs}$, $b > 0$ – Lorenz system

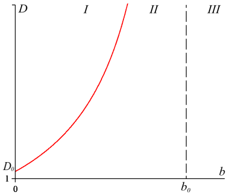
$$z(t) = \int_0^t e^{-b(t-\tau)} x(\tau) y(\tau)\, d\tau \quad \Leftrightarrow \quad \frac{dz}{dt} = xy - bz, \quad z(0) = 0$$

$$J(s) = e^{-bs}(2\sigma - b)/2; \quad 2B(t) = \sigma(D-1) - \int_0^t J(t-\tau) x^2(\tau)\, d\tau$$

$b > 2\sigma \Rightarrow B(t) > 0$ for all t $\quad$ $b < 2\sigma \Rightarrow B(t) > 0$ for some t



Stationary points: $\left(\pm\sqrt{b(D-1)}, \pm\sqrt{b(D-1)}, D-1\right)$.



$b_0 = \sigma - 1$

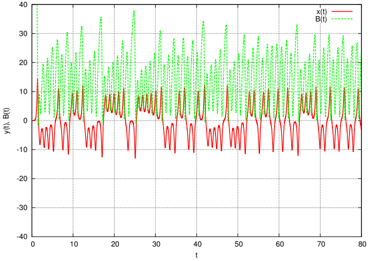
Note that $b > 2\sigma \Rightarrow b > \sigma - 1$.

I – instability points;

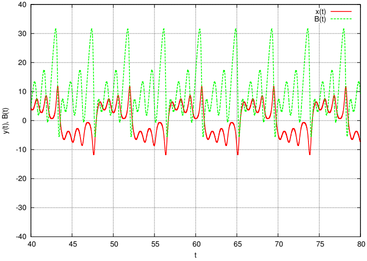
II, III – stability points.

Numerical simulation: $\sigma = 3.37$

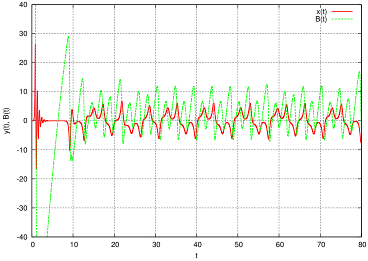
$D = 30$; $b = 1.3$



$D = 60$; $b = 0.5$



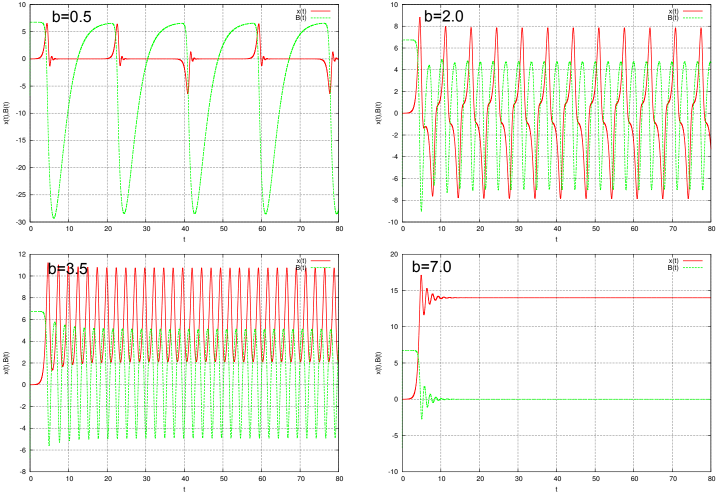
$D = 80$; $b = 0.1$



$K(s) = s \cdot e^{-bs}$, $b > 0$ – delay of feedback

$$J(s) = e^{-bs}\frac{(2\sigma - b)s + 1}{2}; \quad 2B(t) = \sigma(D-1) - \int_0^t J(t-\tau)x^2(\tau)\,d\tau$$

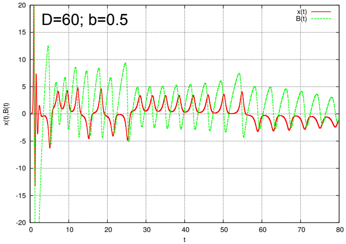
$b < 2\sigma \Rightarrow J(s) > 0$ for all s

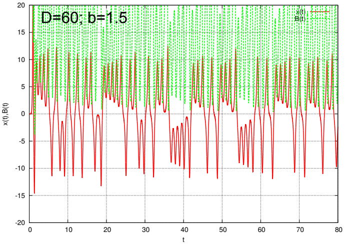


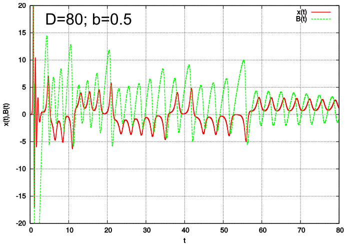
$$K(s) = 1/(1+s)^b, \quad b > 0$$

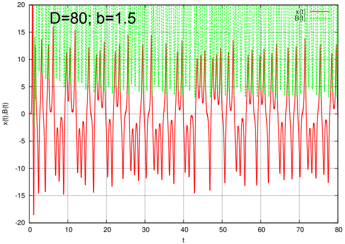
$$J(s) = \frac{2\sigma(1+s) - b}{2(1+s)^{b+1}}; \quad 2B(t) = \sigma(D-1) - \int_0^t J(t-\tau)x^2(\tau)\, d\tau$$

$b < 2\sigma \Rightarrow J(s) > 0$ for all s









Quenching by energy

$$\frac{dx}{dt} = \sigma(y - x),$$
$$\frac{dy}{dt} = (D - z)x - y,$$
$$z(t) = \int_0^t K(t - \tau)\left[x^2(\tau) + y^2(\tau)\right]\, d\tau.$$

Oscillator???

$$\frac{d^2x}{dt^2} + (1+\sigma)\frac{dx}{dt} + \underbrace{\frac{K(0)}{2}x^3 - [\sigma(D-1) - w(t)]\,x}_{\frac{\partial U}{\partial x}} + L(t) = 0,$$

$$w(t) = 2\int_0^t J(t-\tau)x^2(\tau)\, d\tau,$$
$$J(\cdot) = K'(\cdot) + \sigma K(\cdot),$$

$$L(t) = \frac{x}{\sigma}\int_0^t K(t-\tau)\left(\frac{dx(\tau)}{d\tau}\right)^2 d\tau,$$

with initial conditions $x(0) = x_0, x'(0) = \sigma(y_0 - x_0)$.

Conclusions

- $\alpha\Omega$-dynamo system can be considered as an hereditary oscillator
- Different dynamo regimes can be obtained by varying the parameters of the model and the quenching kernel – regular oscillations, chaotic regimes, stable generations, vacillations, dynamo-bursts

THANK YOU